PHYSICAL REVIEW E VOLUME 49, NUMBER 4 APRIL 1994

Eigenvector statistics in the crossover region between Gaussian orthogonal and unitary ensembles

Hans-Jürgen Sommers
Fachbereich Physik, Universität GH Essen, D-45117 Essen, Germany

Shinji Iida
Faculty of Science, Ehime University, Matsuyama 790, Japan
(Received 14 October 1993)

We give a general framework for the joint probability density of an eigenvalue and the corresponding eigenvector. This we exactly determine for random Hamiltonians of the form $H = S + i\alpha A$ where S (A) are symmetric (antisymmetric) N-dimensional matrices whose elements are normally distributed. The random matrices H represent the Gaussian ensemble intermediate between orthogonal ($\alpha = 0$) and unitary ($\alpha = 1$). In the limit of $N \to \infty$, we give the explicit form of the probability density of one component of an eigenvector in the crossover region, $\alpha^2 = O(1/N)$.

PACS number(s): 05.45.+b, 02.50.−r, 24.60.Ky

Random matrices [1], i.e., matrices whose elements are random variables with given probability laws, have been used to describe statistical properties of various quantum systems; random matrices were originally introduced to model highly excited states of complex nuclei [2]. Recent advances in experimental techniques of condensed matter physics have rendered possible the fabrication of submicrometer electronic devices. In such mesoscopic systems there appear fluctuations due to quantum interference effects which are well described by proper random matrix models [3]. In general, it is believed that quantum systems whose classical analogs behave chaotically show universal fluctuation properties which are characteristic of those of random matrix models [4,5].

Various studies indicate that, in the limit of large matrix dimension, correlations of a few eigenvalues and eigenvectors show universal behavior; they depend only on the overall symmetry requirements that a matrix should satisfy and are independent of all other details of the distribution of individual matrix elements. For the sake of theoretical convenience, the Gaussian form of the distribution is usually employed. Three universality classes of random matrices are known from the analysis of symmetries: Time reversal invariant systems are described by the ensemble of real symmetric matrices known as the Gaussian orthogonal ensemble (GOE). On the other hand, the Gaussian unitary ensemble (GUE), the ensemble of Hermitian matrices with equally probable real and imaginary parts, is appropriate to systems with broken time reversal invariance. If spin dependent interactions are important in the time reversal invariant system, the relevant ensemble is the Gaussian symplectic ensemble (GSE) which we will not discuss in this paper. There are also many problems requiring the study of ensembles intermediate between these three classes. Among them, of particular interest are the transitions to the unitary ensemble. For example, in mesoscopic devices, magnetic fields are common experimental tools and play the role of a small perturbation breaking time reversal invariance and drive the transition to the unitary ensemble. In this paper, we consider a GOE-GUE crossover represented by the following random matrix ensemble:

$$H = S + i\alpha A, \tag{1}$$

where S (A) are N-dimensional real symmetric (antisymmetric) statistically independent random matrices. Independent matrix elements of S and A obey the Gaussian distribution with the same variance λ:

$$\langle S_{i\le j}^2 \rangle = (1+\delta_{ij})\lambda, \quad \langle A_{i<j}^2 \rangle = \lambda \tag{2}$$

where the angular bracket means the ensemble average. The parameter α drives the crossover: $\alpha = 0$ (1) corresponds to the GOE (GUE).

Pandey and Mehta studied the eigenvalue statistics of this ensemble and got explicit forms of energy correlation functions [6]. The same crossover has recently been studied for small disordered metals with the help of the supersymmetry method [7]. Less is known for the eigenvector statistics. Even the simplest correlation function, the probability density of one component of an eigenvector, is not available in the crossover region. In the limit of $N \to \infty$, this probability density takes simple forms for the two extreme cases: $\propto e^{-x/2}/\sqrt{x}$ for the GOE and $\propto e^{-x}$ for the GUE where x is proportional to the absolute square of any one component of an eigenvector. Until now, however, there is no explicitly known family of interpolating functions between the above two limits. Since both of the above forms are expressed by a χ_ν^2 distribution of degree ν with $\nu = 1$ for the GOE and $\nu = 2$ for the GUE, a χ_ν^2 with ν continuously varying between 1 and 2 has been frequently used for this purpose. On the other hand, Życzkowski and Lenz (ZL) proposed a different form of the one-parameter family of interpolating functions [8]. Lacking the rigorous derivation, it is not clear how to relate the crossover driving parameter [α in Eq. (1)] with the one in the interpolating function in the above two conjectures. We feel that the basic assumptions of ZL are wrong: due to gauge

freedom the distribution of the eigenvector component can only depend on its modulus and therefore real and imaginary parts cannot be distributed independently according to two different Gaussians. The present stage of theoretical understanding apparently requires more work on the eigenvector statistics in the crossover region. We also mention that such a crossover function is directly concerned with the recent numerical experiments on the conductance through quantum dots under magnetic fields [9]. There the crossover function can be detected from the distribution of maxima of the conductance. Generally it determines the distribution of transition strengths from a fixed state to an eigenstate of the Hamiltonian.

In this paper, we give a formula for the joint probability density of an eigenvalue and the corresponding eigenvector, Eq. (3). The formula is quite general and valid even for general non-Hermitian matrices. For the ensemble intermediate between orthogonal and unitary Eq. (1), we give an expression of this joint probability density for any finite N, Eq. (7). With the help of this result we can give the exact interpolating form of the probability density of one component of an eigenvector in the limit of $N \to \infty$, Eq. (15). This form is also different from previous conjectures: the above mentioned χ^2_ν and the form presented in Ref. [8]. In the following we outline a derivation of our main results focusing on the probability density of one eigenvector component; details will be given elsewhere.

We may obtain the joint probability density of finding an eigenvalue E and the corresponding eigenvector $\{\psi_i\}$ normalized to unity in a volume element $d^2E d^2\psi_1 \cdots d^2\psi_N$ as

$$P(E;\psi_1,\ldots,\psi_N) = \frac{1}{N\pi}\delta\left(\sum_{i=1}^{N}|\psi_i|^2 - 1\right)\left\langle\left|\frac{d}{dE}\mathrm{Det}(H-E)\right|^2\prod_{i=1}^{N}\delta^2\left(\sum_{j=1}^{N}H_{ij}\psi_j - E\psi_i\right)\right\rangle . \tag{3}$$

The formula is valid for any (not necessarily Hermitian) complex N-dimensional matrix H. For the somewhat lengthy derivation, here we can only make some sketchy remarks. We first fix the component $\psi_1 \neq 0$ and consider the Jacobian given by the constraints of the eigenvalue equation. Then we go over to normalized variables and finally we include ψ_1 among the independent variables multiplying by the δ function for the normalization. The phase of ψ_1 is free and therefore chosen equally distributed. This leads to the factor $1/\pi$. Strictly speaking, the formula is valid for nondegenerate eigenvalues, i.e., we assume the degenerate cases being of measure zero. The factor $1/N$ arises because there are N different eigenvalues. For a Hermitian Hamiltonian, P is proportional to $\delta(\mathrm{Im}E)$ implying that the eigenvalues are real. One may easily check the proper normalization of P. Expressing the components ψ_i with the help of the eigenvalue equation by one of them, say ψ_1, the corresponding δ function in the product in (3) is proportional to $\delta^2(\mathrm{Det}(H-E))$. Therefore the complex energy integration may be carried out, which cancels the strange Jacobian in (3) and leads to a sum over all eigenvalues.

The transition ensemble Eq. (1) is still invariant under real orthogonal transformations. Therefore, as manifestly seen from Eq. (3), P may depend only on the following orthogonal invariants:

$$\sum_{i=1}^{N}\psi_i^2,\ \sum_{i=1}^{N}(\psi_i^*)^2,\ \sum_{i=1}^{N}|\psi_i|^2 . \tag{4}$$

Since P cannot depend on a global phase of $\{\psi_i\}$, we may write P as

$$P = \delta(\textstyle\sum|\psi_i|^2 - 1)\delta(\mathrm{Im}E)f(E;\cos^2\gamma) \tag{5}$$

with a function f depending on E and

$$\cos^2\gamma = \left|\sum\psi_i^2\right|^2 . \tag{6}$$

Therefore in the course of calculating P expressed by f we are free to choose $\psi_3 = \psi_4 = \cdots = \psi_N = 0$ and $\psi_2/\psi_1 = e^{i\gamma}$. Then due to normalization it follows $|\psi_1|^2 = |\psi_2|^2 = 1/2$.

We are now able to perform the averaging over the first two rows and columns of H exactly. The last part of integration may be done with the help of the expression of determinants by Grassmann variables. The exact result is

$$f(E,\cos^2\gamma) = \frac{2^N \exp\left[-\frac{E^2}{2\lambda}\left(1-\frac{1}{\Lambda}\right)\right]}{N\pi\alpha^{N-1}(2\pi\lambda)^{N-1/2}D^{(N-2)/2}\Lambda^{1/2}}\left\{\frac{\lambda}{\Lambda}+\left(\frac{E}{\Lambda}\right)^2 - \frac{2E\lambda(1+\alpha^2)}{\Lambda D}\frac{\partial}{\partial E}\right.$$
$$\left. + \left(\frac{2\lambda(1-\alpha^2)\cos\gamma}{D}\right)^2\frac{\partial}{\partial\lambda(1-\alpha^2)} + \left(\frac{2\lambda(1+\alpha^2)}{D}\right)^2\left(\frac{1}{2}\frac{\partial^2}{\partial E^2} + \frac{\partial}{\partial\lambda(1+\alpha^2)}\right)\right\} Z_{N-2} \tag{7}$$

with determinants resulting from Gaussian integrations

$$\Lambda = 2 + \sin^2\gamma\ \left(\frac{1}{\alpha^2} - 1\right) , \tag{8}$$

$$D = 4 + \sin^2\gamma\ \left(\frac{1}{\alpha} - \alpha\right)^2 , \tag{9}$$

and Z_N is given by

$$Z_N = \langle|\mathrm{Det}(H-E)|^2\rangle = \left(-\lambda(1+\alpha^2)\frac{\partial}{\partial\omega}\right)^N \frac{\times\exp\left(-\frac{\omega E^2}{(1-\omega)\lambda(1+\alpha^2)}\right)}{[1+\omega(\frac{1-\alpha^2}{1+\alpha^2})](1+\omega)\sqrt{1-\omega^2}}\Bigg|_{\omega=0} , \tag{10}$$

which is calculated with the help of Grassmann integration. Considering now the orthogonal and unitary limits $\alpha \to 0$ and $\alpha \to 1$, we get exact alternative expressions for the density of states for any finite N. In these limits essentially the δ function for the normalization remains. In the case of $\alpha = 1$ it is easily seen that the γ dependence disappears. The limit $\alpha \to 0$ is more delicate since it produces the δ function for the relative phases of ψ_i.

It is easy to integrate f over energy and we obtain the joint probability density of eigenvector components. This does not depend on the energy scale $\sqrt{\lambda}$. Further on we are able to integrate over a number of variables $d^2\psi_{M+1} \cdots d^2\psi_N$ and get the reduced probability density of components $\{\psi_{i=1\cdots M}\}$. Introducing

$$\sum_{i=M+1}^{N} \psi_i^2 = r^2 \cos\phi + ir^2 \sin\phi \cos\theta \ , \tag{11}$$

$$\sum_{i=M+1}^{N} |\psi_i|^2 = r^2 \ , \tag{12}$$

with $0 \le r < \infty$, $0 \le \phi, \theta \le \pi$, we see the integration measure proportional to

$$r^{2N-2M-1} \sin^{N-M-2}\theta \ \sin^{N-M-1}\phi \ \ dr d\theta d\phi \ . \tag{13}$$

Considering now the case $M = 1$ we get the probability density of one component of eigenvector $P_1(|\psi_1|^2)$. Of particular interest is the limit $N \to \infty$. To get a finite limit, we change the normalization as $\sum_{i=1}^{N} |\psi_i|^2 = N$. Then the average becomes $\langle |\psi_1|^2 \rangle = 1$. If α is finite we find $P_1(x) \propto e^{-x}$, and for $\alpha = 0$ we find $P_1(x) \propto e^{-x/2}/\sqrt{x}$, the Porter-Thomas distribution. Knowing the exact result Eq. (7), we are able to get the probability density $P_1(x) = P_1(x, \epsilon)$ in the crossover region where the properly scaled parameter ϵ is defined as

$$\epsilon = 2\alpha^2 N \ . \tag{14}$$

We finally obtain

$$P_1(x,\epsilon) = \frac{\epsilon e^{\epsilon}}{\pi^2} \int_0^{\pi} d\theta d\phi \frac{\exp\left(-\frac{\epsilon + x(1-\cos\phi)}{\sin^2\phi \sin^2\theta}\right)}{\sin^4\theta \sin^3\phi} \ . \tag{15}$$

This result essentially follows from the D dependence of Eq. (7) and the measure Eq. (13). In this limit only the first term in the curly bracket in Eq. (7) contributes. The integration over θ can be done and leads to a nonelementary function. P_1 is normalized according to $\int d^2\psi_1 P_1(|\psi_1|^2, \epsilon) = 1$. Note that we have also exactly $\int d^2\psi_1 |\psi_1|^2 P_1(|\psi_1|^2, \epsilon) = 1$. With introduction of the scaled parameter ϵ, N has disappeared from the result and $x = |\psi_1|^2$ varies from 0 to ∞. Similarly ϵ varies between 0 and ∞. Now the orthogonal limit is $\epsilon \to 0$ and the unitary limit is $\epsilon \to \infty$. Formula (15) includes both limits exactly.

The value of $P_1(0, \epsilon)$ can be expressed by an error function

$$P_1(0,\epsilon) = \frac{1}{\pi}\left(1 + \int_1^{\infty} dy e^{-\epsilon(y^2-1)}\right) \ . \tag{16}$$

It increases like $1/(2\sqrt{\pi\epsilon})$ for $\epsilon \to 0$. It is easily seen that the first derivative with respect to x at $x = 0$ is given by $P_1^{(1)}(0,\epsilon) = -3P_1(0,\epsilon)/(2\epsilon) - 1/\pi$, which increases even more rapidly as $\epsilon \to 0$. This behavior comes from the fact that at $\epsilon = 0$, $P_1(x, 0)$ is proportional to $e^{-x/2}/\sqrt{x}$ and can no longer be expanded in powers of x. For $\epsilon \to \infty$ it goes to the GUE result $-1/\pi$. We may also give the variance of $x = |\psi_1|^2$ expressed in terms of an exponential integral:

$$\langle x^2 \rangle - \langle x \rangle^2 = 2 - \epsilon e^{\epsilon} E_1(\epsilon) \ . \tag{17}$$

It interpolates between 2 for the GOE and 1 for the GUE. For $\epsilon \to 0$ it goes like $2 + \epsilon \ln \epsilon$.

In order to compare with previous conjectures [8], in Fig. 1 we plot the probability density of $\ln x$

$$W(\ln x) = \pi x P_1(x, \epsilon) \tag{18}$$

for various parameters $\ln \epsilon = -\infty, -1, 0, 1, +\infty$. We find significant differences to both conjectures proposed in Ref. [8]. In Fig. 2 we plot the density of $\ln x$ for $\epsilon = 1$ compared to the conjectures with the same variance given by Eq. (17). This yields for the variance $\langle x^2 \rangle - \langle x \rangle^2 = 1.40$, corresponding to $\nu = 2/1.4$ for χ^2_ν and $b = 1.225$ for Życzkowski and Lenz. Actually the new conjecture by ZL yields a rather good approximation. The maxima are slightly shifted to the left compared with the exact result (15) and in accordance with numerical simulations of those authors on a kicked top [8]. One may ask why the ZL distribution does yield such a good approximation. Actually this distribution appears earlier in the literature [12] and also if one considers the distribution of the squared modulus of one off-diagonal element of our Hamiltonian (1): it is exactly a ZL distribution. In a different model for the magnetoconductance in the strongly localized regime Meir and Entin-Wohlman [13] assume that the overlap amplitudes between different sites are directly given by a random matrix of type (1) and only the distribution of one overlap enters the further calculation, which in that case is exactly a ZL distribution varying on the scale $\alpha \sim 1$. Our theory determines the eigenvector distribution (15), which starts from a random Hamiltonian (1) and relates both via the parameter α. There is no need for fitting any parameter. In concrete physical situations α can be expressed by the magnetic

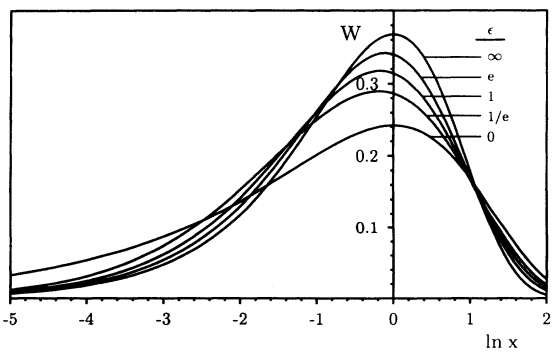

FIG. 1. Density of $\ln x = \ln |\psi_1|^2$ for $\ln \epsilon = -\infty, -1, 0, +1, +\infty$ according to Eqs. (15) and (18).

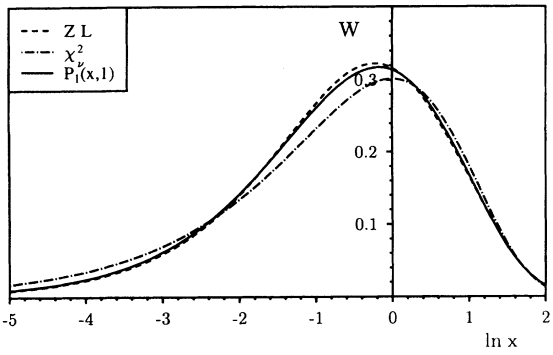

FIG. 2. Density of $\ln x$ for given variance $\langle x^2\rangle - \langle x\rangle^2 = 1.40$ in comparison with $W(\ln x)$ for χ^2_ν and the result of Życzkowski and Lenz (ZL) with parameters $\epsilon = 1$, $\nu = 2/1.4$, $b = 1.225$, respectively. Of course, in all three cases $\langle x\rangle = 1$.

field. As in the case of level spacing the crossover from GOE to GUE occurs already on the scale $\alpha \sim N^{-1/2}$. Comparing with the numerical results of Jalabert *et al.* [9] requires further integrations. Here we provide only the exact expression (15) for the distribution of one eigenvector component, which is central in their paper. However, our results make it possible to generalize the analysis of these authors to a continuous change of the time-reversal breaking magnetic field. The above mentioned reduced densities may be used to consider the many channel case.

We conclude with a remark on the general method (3). It can easily be applied to consider complex [10] or real asymmetric [11] normally distributed matrices. In the case of Hermitian matrices it seems not necessary to allow for general complex energies. However, it is very convenient since due to this formulation the Jacobian is automatically non-negative and therefore allows for a Grassmann representation. Usually one has to take the absolute value of the Jacobian and then a representation in Grassmann variables is not possible. The famous paper on random magnetic fields, supersymmetry and negative dimensions [14] suffers from this deficiency. Here we have found a rigorous way to circumvent this difficulty.

We thank Fritz Haake for discussions. S.I. acknowledges a stay at SFB 237 "Unordnung und grosse Fluktuationen."

[1] M.L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
[2] T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, and S.S.M. Wong, Rev. Mod. Phys. **53**, 385 (1981).
[3] A.D. Stone, P.A. Mello, K.A. Muttalib, and J.-L. Pichard, *Mesoscopic Phenomena in Solids* (North-Holland, Amsterdam, 1991), p. 369.
[4] O. Bohigas, in *Chaos and Quantum Physics*, Les Houches Session LII, edited by M. Gianoni and A. Voros (North-Holland, Amsterdam, 1989).
[5] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1991).
[6] A. Pandey and M.L. Mehta, Commun. Math. Phys. **87**, 449 (1983); M.L. Mehta and A. Pandey, J. Phys. A **16**, 2655 (1983).
[7] A. Altland, S. Iida, and K.B. Efetov (unpublished).
[8] K. Życzkowski and G. Lenz, Z. Phys. B **82**, 299 (1991).
[9] R.A. Jalabert, A.D. Stone, and Y. Alhassid, Phys. Rev. Lett. **68**, 3468 (1992).
[10] J. Ginibre, J. Math. Phys. **6**, 440 (1965).
[11] N. Lehmann and H.-J. Sommers, Phys. Rev. Lett. **67**, 941 (1991).
[12] U. Sivan, O. Entin-Wohlman, and Y. Imry, Phys. Rev. Lett. **60**, 1566 (1988).
[13] Y. Meir and O. Entin-Wohlman, Phys. Rev. Lett. **70**, 1988 (1993).
[14] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).